Journal of Engineering Mathematics, Vol. 3, No. 4, October 1969
Wolters-Noordhoff Publishing – Groningen
Printed in the Netherlands

The Torsional Rigidity of a Cylinder of Curvilinearly Aeolotropic Material

E. E. JONES

University Institute of Science and Technology, Cardiff, Wales

(Received April 2, 1969)

SUMMARY

The problem of the simple torsion of a cylinder possessing aeolotropy defined relative to an orthogonal curvilinear coordinate net is formulated in terms of a stress function. The method of solution of the boundary-value problem for this stress function is indicated, and the torsional rigidity is obtained for both a solid and a hollow elliptic cylinder with a particular type of curvilinear aeolotropy.

1. Introduction

In a recent note [1] it has been shown that the problem of the small twist of a cylinder of general cross-section formed from curvilinearly aeolotropic material exhibiting certain elastic symmetry properties reduces to that of the classical Saint-Venant torsion problem, in that a warping function can be introduced which is determined as the solution of a boundary-value problem of the Neumann type. A feature of the problem is that the elastic aeolotropy is referred to a set of curves one family of which is parallel to the generators of the cylinder, and two other families of curves which are located parallel to the right sections of the cylinder and form an identical orthogonal net in each section. It has been found possible to extend the solution of the basic torsion problem to include a greater range of material symmetries. To this end it has been found convenient to set up the elastic stress-strain relations by means of the tensor analytic approach given by Adkins [2] for the finite torsion of a circular cylinder, due emphasis being given to the fact that the problem considered in the present article refers to the infinitesimal torsional deformation of a cylinder of general cross-sectional geometry.

It has been pointed out by Lekhnitskii [3] that materials of the curvilinearly aeolotropic type may exist as a result of processes involving the extrusion of wires and the manufacture of pipes, and in prismatic structures involving a large number of similar elements.

2. The Warping Function

If the curves defining the material aeolotropy of an elastic body form an orthogonal net defined by the curvilinear coordinates θ^i $(i=1, 2, 3)$, then [2] the stress tensor τ^{ij} $(i, j=1, 2, 3)$ and the infinitesimal strain tensor γ_{ij} $(i, j=1, 2, 3)$, both referred to the θ^i coordinate system, are related by

$$\tau^{ij} = \tfrac{1}{2}\left(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}}\right). \tag{2.1}$$

Here W is the strain energy function for the material, and for curvilinearly aeolotropic material W is a function only of the physical components $\gamma_{(ij)}$ of the strain tensor γ_{ij}, and

$$\gamma_{(ij)} = \gamma_{(ji)} = \gamma_{ij}/(g_{ii}g_{jj})^{\frac{1}{2}}\,, \quad \text{(no sum on } i, j) \tag{2.2}$$

when the curves defining the aeolotropy are orthogonal. It follows from (2.1) and (2.2) that

$$\tau^{ij} = \frac{1}{2(g_{ii}g_{jj})^{\frac{1}{2}}}\left(\frac{\partial W}{\partial \gamma_{(ij)}} + \frac{\partial W}{\partial \gamma_{(ji)}}\right), \quad \text{(no sum on } i, j) \tag{2.3}$$

when W is expressed as a symmetrical function of $\gamma_{(ij)}$ and $\gamma_{(ji)}$.

In the subsequent analysis it will be assumed that a repeated index implies summation over all the values of that index.

For small deformations of the material the strain energy function W can be assumed to be a quadratic function of $\gamma_{(ij)}$, so that

$$W = \tfrac{1}{2} E^{ijkl} \gamma_{(ij)} \gamma_{(kl)} \,, \tag{2.4}$$

where E^{ijkl} $(i, j, k, l = 1, 2, 3)$ are constants, and without loss in generality

$$E^{ijkl} = E^{klij} = E^{jikl} = E^{ijlk} \,.$$

From (2.3) and (2.4) it follows that

$$\tau^{ij} = E^{ijkl} \gamma_{(kl)} / (g_{ii} g_{jj})^{\frac{1}{2}} \quad \text{(no sum on } i, j)$$

and further by use of (2.2)

$$\tau^{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} \mathrm{E}^{ijkl} \gamma_{kl} / (g_{ii} g_{jj} g_{kk} g_{ll})^{\frac{1}{2}} \tag{2.5}$$

with no sum on i, j.

If the elastic body is cylindrical in form rectangular Cartesian coordinates x^i $(i = 1, 2, 3)$ can be introduced with the x^1, x^2 axes located in a typical section S of the cylinder and the x^3 axis in a direction parallel to the generators of the cylinder through a base point O in S. The equations of the curves defining the aeolotropy are considered in the parametric form

$$x^1 = x^1(\theta^1, \theta^2), \; x^2 = x^2(\theta^1, \theta^2), \; x^3 = \theta^3 \,, \tag{2.6}$$

with a non-vanishing Jacobian J. Alternatively another notation may be introduced, viz.

$$x^1 \equiv x, \; x^2 \equiv y, \; x^3 \equiv z, \; \theta^1 \equiv \xi, \; \theta^2 \equiv \eta \,, \tag{2.7}$$

so that the first two members of (2.6) become

$$x = x(\xi, \eta), \quad y = y(\xi, \eta) \,, \tag{2.8}$$

and we write $x_\xi \equiv \partial x / \partial \xi$, $x_\eta \equiv \partial x / \partial \eta$, etc.

The corresponding covariant metric tensor is defined by

$$g_{ij} = x^k_{,i} x^k_{,j}$$

with $x^k_{,i} \equiv \partial x^k / \partial \theta^i$, and in the notation of (2.7) we have

$$g_{11} = A^2, \; g_{22} = B^2, \; g_{33} = 1, \; g_{ij} = 0 \qquad (i \neq j) \,, \tag{2.9}$$

where

$$A = (x_\xi^2 + y_\xi^2)^{\frac{1}{2}} \,, \quad B = (x_\eta^2 + y_\eta^2)^{\frac{1}{2}} \,, \tag{2.10}$$

and since the curves defining the aeolotropy are orthogonal g_{12} has been equated to zero, implying that

$$x_\xi x_\eta + y_\xi y_\eta = 0 \,. \tag{2.11}$$

Similarly the contravariant tensor g^{ij} has components

$$g^{11} = 1/A^2, \; g^{22} = 1/B^2, \; g^{33} = 1, \; g^{ij} = 0 \qquad (i \neq j) \,. \tag{2.12}$$

By use of (2.11) it is seen that the Jacobian of the transformation (2.6) has the value

$$J = x_\xi y_\eta - x_\eta y_\xi = AB \,, \tag{2.13}$$

and thus for non-vanishing J we must have $A \neq 0$ and $B \neq 0$ in S.

If u_i $(i = 1, 2, 3)$ are the Cartesian components of the displacement referred to the x^i axes, then for small displacements the strain tensor e_{ij} is defined by

$$e_{ij} = \tfrac{1}{2} \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} \right) . \tag{2.14}$$

When the cylinder undergoes simple torsion with uniform small twist θ per unit length of the cylinder the displacement components may be assumed in the form

$$u_1 = -\theta x^2 x^3, \quad u_2 = \theta x^3 x^1, \quad u_3 = \theta\phi(x^1, x^2), \tag{2.15}$$

where $\phi(x^1, x^2) \equiv \phi(x, y)$ is the warping function. In the notation of (2.7), it can be deduced from (2.14) and (2.15) that the only non-zero strain components are

$$e_{23} = \frac{\theta}{2}\left(\frac{\partial\phi}{\partial y} + x\right), \quad e_{31} = \frac{\theta}{2}\left(\frac{\partial\phi}{\partial x} - y\right). \tag{2.16}$$

The corresponding strain components γ_{ij} referred to the θ^i curvilinear axial system formed by the curves defining the aeolotropy are given by

$$\gamma_{ij} = x^k_{,i} x^l_{,j} e_{kl},$$

and from (2.6) and (2.16) the only non-zero components of γ_{ij} are

$$\left.\begin{aligned} \gamma_{13} &= x_\xi e_{13} + y_\xi e_{23}, \\ \gamma_{23} &= x_\eta e_{13} + y_\eta e_{23}. \end{aligned}\right\} \tag{2.17}$$

At this stage it is necessary to specify the elastic symmetry properties of the elastic material. In particular in (2.5) it is assumed that

$$E^{ijk3} = E^{i333} = 0 \qquad (i, j, k = 1, 2), \tag{2.18}$$

and for convenience we write

$$E^{3113} = p, \quad E^{3223} = q, \quad E^{1323} = r. \tag{2.19}$$

The classes of the various possible symmetry systems are referred to by numbers according to their ordering in the list given by Green and Adkins [4], pp. 13–25.

(i) The group $r=0$, $p=q$, as in [1], includes the classes 1–7 of the tetragonal system with the θ^3 axis as basis, the cubic system, the classes 6–12 of the hexagonal system with the θ^3 axis as basis, transverse isotropy about the θ^3 axis, and the isotropic system.

(ii) The group $r=0$, $p\neq q$, includes the rhombic system, the classes 4–7 of the tetragonal system with the θ^1 axis, or the θ^2 axis, as basis, and transverse isotropy about the θ^1 axis or the θ^2 axis.

(iii) The group $r\neq 0$, $p\neq q$, includes the monoclinic system with the θ^3 axis as basis, and the classes 3–5 of the hexagonal system with the θ^2 axis as basis.

On substitution from (2.9), (2.18) and (2.19) into (2.5) it follows that the only non-zero stress components are

$$\left.\begin{aligned} \tau^{13} &= \frac{2p}{A^2}\gamma_{13} + \frac{2r}{AB}\gamma_{23}, \\ \tau^{23} &= \frac{2r}{AB}\gamma_{13} + \frac{2q}{B^2}\gamma_{23}. \end{aligned}\right\} \tag{2.20}$$

It is also possible to express τ^{13} and τ^{23} in terms of the warping function $\phi(\xi, \eta) \equiv \phi(x[\xi, \eta], y[\xi, \eta])$ by use of (2.16) and (2.17) in (2.20) leading to

$$\left.\begin{aligned} \tau^{13} &= \frac{p\theta}{A^2}(\phi_\xi - yx_\xi + xy_\xi) + \frac{r\theta}{AB}(\phi_\eta - yx_\eta + xy_\eta), \\ \tau^{23} &= \frac{r\theta}{AB}(\phi_\xi - yx_\xi + xy_\xi) + \frac{q\theta}{B^2}(\phi_\eta - yx_\eta + xy_\eta). \end{aligned}\right\} \tag{2.21}$$

The condition that the strain energy function W of (2.4) is positive definite restricts the values of p, q, r. In fact from (2.2), (2.4) and (2.5) it is seen that

$$W = \tfrac{1}{2}\tau^{ij}\gamma_{ij},$$

hence from (2.20)

$$\begin{aligned} W &= 2\{p(B\gamma_{13})^2 + 2rAB\gamma_{13}\gamma_{23} + q(A\gamma_{23})^2\}/A^2B^2 \\ &= 2\{(pB\gamma_{13} + rA\gamma_{23})^2 + (pq - r^2)(A\gamma_{23})^2\}/pA^2B^2 . \end{aligned}$$

Sufficient conditions for W to be positive definite are therefore $p > 0$ and $pq - r^2 > 0$, and these imply that $q > 0$.

3. The Stress Function

The equilibrium conditions [5] which restrict the values of the stress components τ^{ij} are

$$\tau^{ij}|_j = 0 , \qquad (i,j = 1, 2, 3) \tag{3.1}$$

where the single bar denotes covariant differentiation with respect to the θ^i coordinate system, so that, since τ^{13}, τ^{23}, the only non-zero stress components, are functions of θ^1 and θ^2 only, and the Christoffel symbols $\Gamma^i_{j3} = 0$ $(i, j = 1, 2, 3)$, then the only non-trivial equation occurs when $i = 3$, and this takes the form

$$\frac{\partial \tau^{3\alpha}}{\partial \theta^\alpha} + \Gamma^j_{\alpha j}\tau^{3\alpha} = 0 \qquad (\alpha = 1, 2) . \tag{3.2}$$

But from [5] it is known that

$$\Gamma^j_{\alpha j} = \frac{1}{g^{\frac{1}{2}}}\frac{\partial g^{\frac{1}{2}}}{\partial \theta^\alpha},$$

where g is the determinant $|g_{ij}|$, and from (2.9)

$$g = A^2B^2 ,$$

hence (3.2) may be written as

$$\frac{1}{AB}\frac{\partial}{\partial \theta^\alpha}(AB\tau^{3\alpha}) = 0 .$$

But $A \neq 0$ and $B \neq 0$, hence in terms of ξ, η we have

$$\frac{\partial}{\partial \xi}(AB\tau^{13}) + \frac{\partial}{\partial \eta}(AB\tau^{23}) = 0 , \tag{3.3}$$

which is the partial differential equation that restricts τ^{13} and τ^{23}.

Now (3.3) is satisfied identically when

$$\tau^{13} = \psi_\eta/AB , \quad \tau^{23} = -\psi_\xi/AB , \tag{3.4}$$

where $\psi(\xi, \eta)$ is referred to as the stress function, and is arbitrary in form.

On substitution from (2.21) into (3.4), and solving for ϕ_ξ and ϕ_η, we have

$$\begin{aligned} \phi_\eta - yx_\eta + xy_\eta &= (rA\psi_\eta + pB\psi_\xi)/\theta A(r^2 - pq) , \\ \phi_\xi - yx_\xi + xy_\xi &= -(qA\psi_\eta + rB\psi_\xi)/\theta B(r^2 - pq) . \end{aligned} \tag{3.5}$$

Again ϕ can be eliminated from (3.5), and with the use of (2.13), leads to

$$p\frac{\partial}{\partial \xi}\left(\frac{B}{A}\frac{\partial \psi}{\partial \xi}\right) + 2r\frac{\partial^2 \psi}{\partial \xi\,\partial \eta} + q\frac{\partial}{\partial \eta}\left(\frac{A}{B}\frac{\partial \psi}{\partial \eta}\right) = 2(r^2 - pq)\theta AB , \tag{3.6}$$

and this is the partial differential equation for the determination of the stress function $\psi(\xi, \eta)$.

The stress components σ_{ij} referred to the Cartesian system of coordinates x^i are related to the stress components τ^{ij} of (2.5) by the relations

$$\sigma_{ij} = x^i_{,k} x^j_{,l} \tau^{kl} ,$$

and from (2.6), (2.7) the only non-zero components are

$$\left.\begin{aligned} \sigma_{13} &= x_\xi \tau^{13} + x_\eta \tau^{23} , \\ \sigma_{23} &= y_\xi \tau^{13} + y_\eta \tau^{23} . \end{aligned}\right\} \tag{3.7}$$

Here τ^{13}, τ^{23} are given by (3.4), so that

$$\begin{aligned} \sigma_{13} &= (x_\xi \psi_\eta - x_\eta \psi_\xi)/AB , \\ \sigma_{23} &= (y_\xi \psi_\eta - y_\eta \psi_\xi)/AB . \end{aligned} \tag{3.8}$$

It is evident from (2.11) and (2.13) that

$$x_\xi = Ay_\eta/B , \quad x_\eta = -By_\xi/A , \tag{3.9}$$

so that on expressing ψ_ξ, ψ_η in terms of ψ_x, ψ_y, where $\psi_x \equiv \partial\psi/\partial x$, $\psi_y \equiv \partial\psi/\partial y$, and using (3.9), we have

$$\sigma_{13} = \psi_y , \quad \sigma_{23} = -\psi_x . \tag{3.10}$$

The boundary conditions [5] on C, the boundary of the cross-section S, are

$$n_i \tau^{ij} = 0 , \tag{3.11}$$

where n_i is the unit normal vector to C in the plane of S. Alternatively these conditions may be expressed relative to the x^i Cartesian coordinate system by use of (3.10), and they reduce to the condition

$$\psi = constant \tag{3.12}$$

on C, by a method similar to that of [6, p.129], where this constant takes different values in general on the various closed curves constituting C. For convenience it is usual to assume that $\psi = 0$ on the external boundary of the section S.

It may also be shown, from (3.10), (3.12) and use of Green's theorem, that the resultant force applied at the end cross-section of the cylinder is zero in magnitude.

4. The Torsional Rigidity

If the moment of the torsional couple applied to the end cross-sections of the cylinder has magnitude τ, then over the cross-section S in the x–y plane, by definition

$$\tau = \int_S (x\sigma_{23} - y\sigma_{13}) dx\, dy , \tag{4.1}$$

where σ_{13}, σ_{23} are given by (3.8) in terms of ψ_ξ, ψ_η. On integration over the corresponding region S', defined by the transformation (2.6), (2.7), and using (3.8), we have

$$\tau = \int_{S'} \{(xy_\xi - yx_\xi)\psi_\eta - (xy_\eta - yx_\eta)\psi_\xi\} d\xi\, d\eta , \tag{4.2}$$

since the Jacobian of the transformation is given by (2.13).

From (3.9) a further reduction in the form of (4.2) is possible. In fact

$$\tau = -\int_{S'} \left\{\frac{A}{B}(xx_\eta + yy_\eta)\psi_\eta + \frac{B}{A}(xx_\xi + yy_\xi)\psi_\xi\right\} d\xi\, d\eta ,$$

or

$$\tau = -\tfrac{1}{2}\int_{S'} \left\{\frac{A}{B} P_\eta \psi_\eta + \frac{B}{A} P_\xi \psi_\xi\right\} d\xi\, d\eta , \tag{4.3}$$

where $P=x^2+y^2$, and $P_\xi \equiv \partial P/\partial\xi$, $P_\eta \equiv \partial P/\partial\eta$ as usual. Equation (4.3) is that from which the torsional rigidity of the cylinder can be determined when ψ is known.

5. Curves of Special Geometry

The curves defining the aeolotropy of the material forming the cylinder with equations of the type (2.8) are restricted so that

$$z = z(\zeta)\,, \tag{5.1}$$

where $z=x+iy$ and $\zeta=\xi+i\eta$. Here $z(\zeta)$ is a regular function of the complex variable ζ except possibly at prescribed singular points in S.

In this case the Cauchy–Riemann equations

$$x_\xi = y_\eta\,, \qquad x_\eta = -y_\xi \tag{5.2}$$

are satisfied, so that $x(\xi,\eta)$, $y(\xi,\eta)$ are harmonic functions, and the curves defining the aeolotropy are the level curves $\xi = constant$, $\eta = constant$, forming an orthogonal net in S. It is noted from (2.10) and (5.2) that

$$A = B\,, \tag{5.3}$$

except possibly at prescribed points in S, so that the partial differential equation for the stress function ψ, given by (3.6), reduces to

$$p\,\frac{\partial^2\psi}{\partial\xi^2} + 2r\,\frac{\partial^2\psi}{\partial\xi\,\partial\eta} + q\,\frac{\partial^2\psi}{\partial\eta^2} = -2(pq-r^2)A^2\theta\,. \tag{5.4}$$

It is now possible to introduce associated complex functions

$$\chi = \xi+\mu\eta\,, \qquad \bar{\chi} = \xi+\bar{\mu}\eta\,, \tag{5.5}$$

where μ is a complex constant, and $\bar{\mu}$ is its complex conjugate. The constants μ, $\bar{\mu}$ are chosen to be the roots of the quadratic equation

$$q\lambda^2+2\lambda r+p=0\,, \tag{5.6}$$

so that in particular $\mu=u+iv$, where

$$u=-r/q, \qquad v=(pq-r^2)^{\frac{1}{2}}/q\,. \tag{5.7}$$

In terms of χ, $\bar{\chi}$ equation (5.4) takes the form

$$\frac{\partial^2\psi}{\partial\chi\,\partial\bar{\chi}} = -\tfrac{1}{2}qA^2\theta\,. \tag{5.8}$$

But from (5.2)

$$x_{\xi\xi}+x_{\eta\eta} = y_{\xi\xi}+y_{\eta\eta} = 0\,,$$

then from (2.10) and (5.3) we have

$$\begin{aligned} 2A^2 &= x_\xi^2+y_\xi^2+x_\eta^2+y_\eta^2 \\ &= \frac{\partial}{\partial\xi}(xx_\xi+yy_\xi) + \frac{\partial}{\partial\eta}(xx_\eta+yy_\eta)\,, \end{aligned}$$

or, with $P=x^2+y^2$, it follows that

$$A^2 = \tfrac{1}{4}\left(\frac{\partial^2 P}{\partial\xi^2} + \frac{\partial^2 P}{\partial\eta^2}\right). \tag{5.9}$$

In terms of χ, $\bar{\chi}$ of (5.5), alternatively

$$4A^2 = (1+\mu^2)\frac{\partial^2 P}{\partial \chi^2} + 2(1+\mu\bar{\mu})\frac{\partial^2 P}{\partial \chi \partial \bar{\chi}} + (1+\bar{\mu}^2)\frac{\partial^2 P}{\partial \bar{\chi}^2}, \tag{5.10}$$

so that (5.8) is equivalent to

$$\frac{\partial^2 \psi}{\partial \chi \partial \bar{\chi}} = -\frac{q\theta}{8}\left\{(1+\mu^2)\frac{\partial^2 P}{\partial \chi^2} + 2(1+\mu\bar{\mu})\frac{\partial^2 P}{\partial \chi \partial \bar{\chi}} + (1+\bar{\mu}^2)\frac{\partial^2 P}{\partial \bar{\chi}^2}\right\}. \tag{5.11}$$

The general solution of the partial differential equation (5.11) may be written in the form

$$\psi = \psi_0 + \Psi, \tag{5.12}$$

where ψ_0 is a particular integral of (5.11), and Ψ is a general solution of the corresponding homogeneous partial differential equation

$$\frac{\partial^2 \Psi}{\partial \chi \partial \bar{\chi}} = 0 \tag{5.13}$$

A particular integral of (5.11) is readily shown to be

$$\psi_0 = -\frac{q\theta}{8}\left\{(1+\mu^2)\int \frac{\partial P}{\partial \chi} d\bar{\chi} + (1+\bar{\mu}^2)\int \frac{\partial P}{\partial \bar{\chi}} d\chi + 2(1+\mu\bar{\mu})P\right\},$$

and the general solution of (5.13) has the form

$$\Psi = \tfrac{1}{2}f(\chi) + \tfrac{1}{2}\bar{f}(\bar{\chi}),$$

where $f(\chi)$ is an arbitrary function of χ. It follows that the complete solution of (5.11) may be written as

$$\psi = -\frac{q\theta}{4}\left\{(1+\mu\bar{\mu})P + re(1+\bar{\mu}^2)\int \frac{\partial P}{\partial \bar{\chi}} d\chi\right\} + ref(\chi), \tag{5.14}$$

where $P = x^2 + y^2 = z\bar{z}$, and z, $\bar{z}$ are given by (5.1).

The moment of the torsional couple acting on the cylinder is given by (4.3), and thus by use of (5.3), we find that

$$\tau = -\tfrac{1}{2}\int_{S'}\left(\frac{\partial P}{\partial \eta}\frac{\partial \psi}{\partial \eta} + \frac{\partial P}{\partial \xi}\frac{\partial \psi}{\partial \xi}\right) d\xi\, d\eta. \tag{5.15}$$

But $P = z\bar{z}$ is a function of ζ, $\bar{\zeta}$ from (5.1), and thus

$$\frac{\partial P}{\partial \xi} = \frac{\partial P}{\partial \zeta} + \frac{\partial P}{\partial \bar{\zeta}}, \qquad \frac{\partial P}{\partial \eta} = i\left(\frac{\partial P}{\partial \zeta} - \frac{\partial P}{\partial \bar{\zeta}}\right).$$

Similar relations hold for $\partial\psi/\partial\xi$ and $\partial\psi/\partial\eta$, hence (5.15) becomes

$$\tau = -2re\int_{S'} \frac{\partial P}{\partial \zeta}\frac{\partial \psi}{\partial \bar{\zeta}} d\xi\, d\eta,$$

or, since $P = z(\zeta)\bar{z}(\bar{\zeta})$, then finally

$$\tau = -2re\int_{S'} \bar{z}\frac{\partial z}{\partial \zeta}\frac{\partial \psi}{\partial \bar{\zeta}} d\xi\, d\eta. \tag{5.16}$$

The torsional rigidity of the cylinder can be defined as τ/θ, where θ is the twist per unit length of the cylinder during torsion.

Example 1: the solid elliptic cylinder

It is assumed that the cross-section S of the cylinder has an external boundary C in the form of an ellipse with semi-major axis a and semi-minor axis b. This ellipse is the curve corresponding to $\eta = 0$ in the net defined by the transformation

$$z = x + iy = c\cos(\zeta + i\alpha), \qquad \zeta = \xi + i\eta, \tag{5.17}$$

where c, α are real constants such that $a=c\cosh\alpha$, $b=c\sinh\alpha$. The line of foci corresponds to the value $\eta=-\alpha$.

The transformation (5.17) is conformal inside the curve corresponding to $\eta=0$ except at the points where $\xi=0$, $\eta=-\alpha$ and $\xi=\pi$, $\eta=-\alpha$, and these are the foci of the ellipse. The curves defining the aeolotropy of the material forming the cylinder are assumed to be the level curves of this net, with $\eta=$constant on each curve.

From (5.5) it is possible to write

$$\zeta=\{(\mu-i)\bar{\chi}-(\bar{\mu}-i)\chi\}/(\mu-\bar{\mu}), \tag{5.18}$$

with $\bar{\zeta}$ as its conjugate. Hence in (5.14) the function P may be determined in terms of χ and $\bar{\chi}$ by use of (5.17), (5.18), and integration leads to the expression

$$\psi=\lambda\left[\mu\bar{\mu}\{e^{i(\zeta-\bar{\zeta})-2\alpha}+e^{-i(\zeta-\bar{\zeta})+2\alpha}\}+e^{i(\zeta+\bar{\zeta})}+e^{-i(\zeta+\bar{\zeta})}\right]+\tfrac{1}{2}f(\chi)+\tfrac{1}{2}\bar{f}(\bar{\chi}), \tag{5.19}$$

where $\lambda=qc^2\theta(\mu-\bar{\mu})^2/32\mu\bar{\mu}$.

Along the line of foci of the elliptic cross-section the corresponding value of η is $-\alpha$, so that there

$$\psi=\lambda(2\mu\bar{\mu}+e^{2i\xi}+e^{-2i\xi})+\tfrac{1}{2}f(\xi-\mu\alpha)+\tfrac{1}{2}\bar{f}(\xi-\bar{\mu}\alpha). \tag{5.20}$$

By reference to (5.17) it is noted that the first term in the right hand member of (5.20) is a quadratic function of z, hence it is continuous across the line of foci of the elliptic section of the cylinder. In order to ensure that the complete expression for ψ is continuous across this line of foci it is sufficient to choose

$$f(\chi)=A_0+A_1\{e^{2i(\chi+\mu\alpha)}+e^{-2i(\chi+\mu\alpha)}\} \tag{5.21}$$

in (5.19), where A_0 and A_1 are complex constants.

The physical stress components corresponding to τ^{13}, τ^{23} of (3.4) are v^{13}, v^{23}, where

$$v^{13}=A\tau^{13}, \quad v^{23}=B\tau^{23}. \tag{5.22}$$

But from (2.10) and (5.1)–(5.3) we find that

$$A^2=z'(\zeta)\bar{z}'(\bar{\zeta}),$$

so that by use of (3.4) equation (5.22) gives

$$v^{13}-iv^{23}=\frac{2i}{A}\frac{\partial\psi}{\partial\zeta},$$

and it may readily be verified that v^{13}, v^{23} are finite at the foci of the elliptic cross-section of the cylinder, where the transformation (5.17) ceases to be conformal, i.e. where $z'(\zeta)=0$.

It is also required that the function ψ given by (5.19), (5.21) must satisfy the condition $\psi=0$ on C, where $\eta=0$. It can be proved that this is so for all values of $(0\leqslant\xi\leqslant 2\pi)$ if

$$A_0+\bar{A}_0=-\tfrac{1}{8}qc^2\theta(\mu-\bar{\mu})^2\cosh 2\alpha, \tag{5.23}$$

and

$$A_1=\frac{2\lambda(e^{2i\bar{\mu}\alpha}-e^{-2i\bar{\mu}\alpha})}{e^{2i\alpha(\mu-\bar{\mu})}-e^{-2i\alpha(\mu-\bar{\mu})}}, \tag{5.24}$$

and these values should be used in connection with (5.21).

The moment of the torsional couple applied to the cylinder is given by (5.16), and in this case takes the form

$$\tau=-2re\int_0^{2\pi}\int_{-\alpha}^{0}\bar{z}\frac{dz}{d\zeta}\frac{\partial\psi}{\partial\bar{\zeta}}\,d\xi\,d\eta. \tag{5.25}$$

The functions in the integrand are obtained from (5.17), (5.19), (5.21), (5.23) and (5.24), and on integration it is found that the only non-zero contributions come from terms independent of ξ in the integrand. After some reduction it may be shown that

$$\tau = \frac{\pi q c^4 \theta v^2}{16(u^2+v^2)} \left\{ 4(1-u^2-v^2)\alpha + (u^2+v^2)\sinh 4\alpha + \frac{2v}{u^2+v^2} \frac{\cos 4\alpha u - \cosh 4\alpha v}{\sinh 4\alpha v} \right\}, \quad (5.26)$$

where u, v are defined by (5.7). In an alternative form, with $\gamma = (pq - r^2)^{\frac{1}{2}}$ for convenience, we also have

$$\tau = \frac{\pi c^4 \theta \gamma^2}{16pq} \left\{ 4(q-p)\alpha + p \sinh 4\alpha + \frac{2q\gamma}{p} \frac{\cos(4\alpha r/q) - \cosh(4\alpha\gamma/q)}{\sinh(4\alpha\gamma/q)} \right\}. \quad (5.27)$$

When $r=0$, i.e. the elastic constant $E^{1323}=0$, then the torsional couple reduces to

$$\tau = \frac{\pi q c^4 \theta}{16} \left\{ 4\left(1 - \frac{p}{q}\right)\alpha + \frac{p}{q}\sinh 4\alpha - 2\left(\frac{q}{p}\right)^{\frac{1}{2}} \tanh 2\alpha \left(\frac{p}{q}\right)^{\frac{1}{2}} \right\}, \quad (5.28)$$

and if further $p=q$, i.e. $E^{3113}=E^{3223}$, then in terms of a, b, the semi-axes of the elliptic section

$$\tau = \pi q \theta a^3 b^3/(a^2+b^2), \quad (5.29)$$

which for an isotropic material is a well-known result [6, p. 122].

In (5.27)–(5.29), with $p>0$, $q>0$, $\gamma^2>0$, since (i) τ is continuous for $\alpha \geqq 0$, (ii) $\tau=0$ for $\alpha=0$, and (iii) $d\tau/d\alpha>0$ for $\alpha>0$, then it follows that $\tau>0$ for $\alpha>0$ ($p>0$, $q>0$, $\gamma^2>0$). This must be so since a twist in a certain sense should be the result of a couple applied to the cylinder in the same sense.

In order to indicate how the torsional rigidity varies with changes in the section geometry and in the elastic coefficients of the cylinder, it is found convenient to introduce the dimensionless quantities $\tau^* = 16\tau/\pi q \theta a^4$, $b^* = b/a$, $p^* = p/q$, $r^* = r/q$ into (5.27).

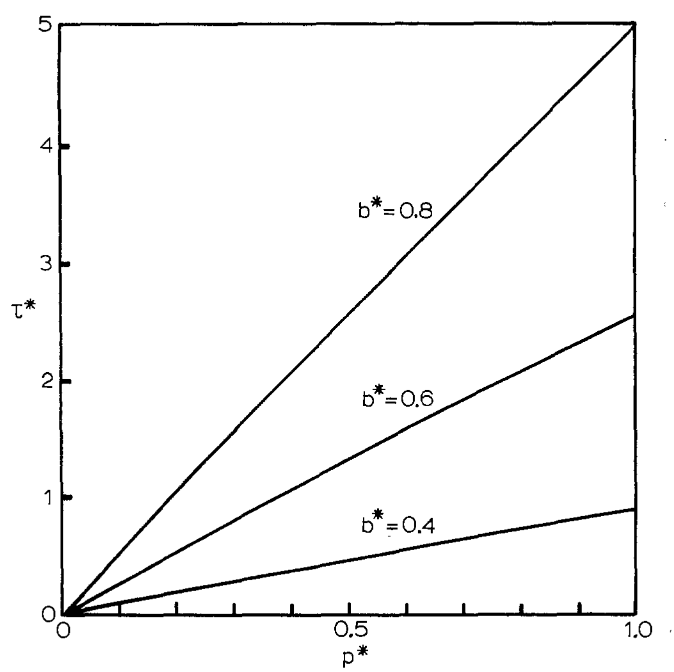

Figure 1. Variation of τ^* with p^* for $r^*=0.1$.

Fig. 1 shows that τ^* increases with p^* for fixed r^* and b^*, the rate of increase being greater for larger values of b^*, i.e. as the sections approach the circular form. Fig. 2 shows that τ^* increases with b^* for fixed p^* and r^*, i.e. as the cylinder section approaches the circular form, and the rate of increase is greater for smaller values of r^*.

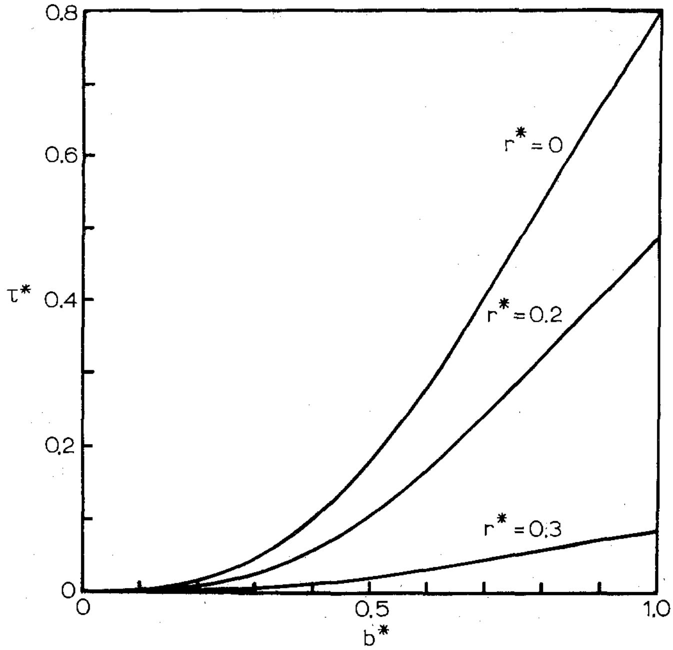

Figure 2. Variation of τ^* with b^* for $p^*=0.1$.

Example 2: the hollow elliptic cylinder

It is assumed that the cylinder has internal and external cross-sectional contours in the form of confocal ellipses, these being the level surfaces $\eta=\alpha_2, \alpha_1$ ($>\alpha_2$) respectively in the net defined by

$$z=c\cos\zeta, \quad \zeta=\xi+i\eta\,, \tag{5.30}$$

where c is a real constant. As in Example 1 integration of (5.14) leads to

$$\psi=\lambda[\mu\bar{\mu}\{e^{i(\zeta-\bar{\zeta})}+e^{-i(\zeta-\bar{\zeta})}\}+e^{i(\zeta+\bar{\zeta})}+e^{-i(\zeta+\bar{\zeta})}]+re\,f(\chi)\,. \tag{5.31}$$

But it is required that $\psi=0$ when $\eta=\alpha_1$ so that

$$\lambda\{\mu\bar{\mu}(e^{-2\alpha_1}+e^{2\alpha_1})+e^{2i\xi}+e^{-2i\xi}\}+re\,f(\chi_1)=0\,, \tag{5.32}$$

where $\chi_1=\xi+\mu\alpha_1$. Again it is required that $\psi=$constant when $\eta=\alpha_2$, so that

$$\lambda\{\mu\bar{\mu}(e^{-2\alpha_2}+e^{2\alpha_2})+e^{2i\xi}+e^{-2i\xi}\}+re\,f(\chi_2)=\text{constant}\,, \tag{5.33}$$

where $\chi_2=\xi+\mu\alpha_2$.

Equations (5.32), (5.33) may be satisfied by choosing

$$f(\chi)=A_0+A_1e^{2i\chi}+A_2e^{-2i\chi}\,,$$

where $\chi=\xi+\mu\eta$, and

$$\tfrac{1}{2}(A_0+A_0)=\text{constant}-2\lambda\mu\bar{\mu}\cosh 2\alpha_2\,,$$

$$A_1=A_2e^{-2i\mu(\alpha_1+\alpha_2)}=\frac{2\lambda(e^{2i\bar{\mu}\alpha_1}-e^{2i\bar{\mu}\alpha_2})}{e^{2i(\mu\alpha_1+\bar{\mu}\alpha_2)}-e^{2i(\mu\alpha_2+\bar{\mu}\alpha_1)}}\,.$$

The torsional couple on the cylinder has moment

$$\tau=-2re\int_0^{2\pi}\int_{\alpha_2}^{\alpha_1}\bar{z}\frac{dz}{d\zeta}\frac{\partial\psi}{d\bar{\zeta}}\,d\xi\,d\eta\,,$$

and reduces on integration to

$$\tau = \frac{c^4\theta\gamma^2}{16pq}\left\{4(q-p)\beta+p(\sinh 4\alpha_1-\sinh 4\alpha_2) + \frac{4q\gamma}{p}\frac{\cos(2r\beta/q)-\cosh(2\gamma\beta/q)}{\sinh(2\gamma\beta/q)}\right\}, \quad (5.34)$$

where $\beta=\alpha_1-\alpha_2$.

In particular if $r=0$, then further

$$\tau = \frac{\pi q c^4\theta}{16}\left\{4\beta\left(1-\frac{p}{q}\right)+\frac{p}{q}(\sinh 4\alpha_1-\sinh 4\alpha_2)-4\left(\frac{q}{p}\right)^{\frac{1}{2}}\tanh\beta\left(\frac{p}{q}\right)^{\frac{1}{2}}\right\}, \quad (5.35)$$

and again if $p=q$ then

$$\tau = \frac{\pi c^4 p\theta}{16}(\sinh 4\alpha_1-\sinh 4\alpha_2-4\tanh\beta), \quad (5.36)$$

which for an isotropic material is a known result, [7, p. 394].

In (5.31)–(5.33) with $p>0$, $q>0$, $\gamma^2>0$, since (i) τ is continuous for $\alpha_1 \geqq \alpha_2$, (ii) $\tau=0$ for $\alpha_1=\alpha_2$, and (iii) $d\tau/d\alpha_1>0$ for $\alpha_1>\alpha_2$, then it follows that $\tau>0$ for $\alpha_1>\alpha_2$ ($p>0$, $q>0$, $\gamma^2>0$), as is to be expected.

Appendix: change of axis of twist

The displacement components u_i as agiven by (2.15), when the cylinder undergoes a twist about the z-axis, are in an alternative notation

$$u_1 = -\theta yz, \quad u_2=\theta xz, \quad u_3=\theta\phi(x,y). \quad (A1)$$

If the twist takes place about a line parallel to the z-axis through the point $(\bar{x}, \bar{y})$ in S, then the corresponding displacement components are

$$\bar{u}_1 = -\theta(y-\bar{y})z, \quad \bar{u}_2=\theta(x-\bar{x})z, \quad \bar{u}_3=\theta\bar{\phi}(x,y), \quad (A2)$$

where $\bar{\phi}(x,y)$ is a possibly different function from $\phi(x,y)$. The corresponding non-zero stress components $\bar{\tau}^{3j}$ and τ^{3j} ($j=1,2$) both satisfy equilibrium equations of the type (3.1) in S, and boundary conditions of the type (3.11) on C.

It follows that the tensor difference functions

$$T^{3j}=\tau^{3j}-\bar{\tau}^{3j} \qquad (j=1,2) \quad (A3)$$

are such that

$$T^{3j}|_j=0 \quad \text{in} \quad S, \quad (A4)$$

and

$$n_j T^{3j}=0 \quad \text{on} \quad C. \quad (A5)$$

It is now possible to introduce the function

$$\Phi=\phi-\bar{\phi}-\bar{y}x+\bar{x}y, \quad (A6)$$

and from (A4) it follows that

$$(\Phi T^{3j})|_j=\Phi|_j T^{3j}=\Phi_{,j}T^{3j} \quad (A7)$$

in S. An application of the tensor form of Gauss' theorem gives

$$\int_S (\Phi T^{3j})|_j dS = \int_C \Phi n_j T^{3j} ds = 0$$

by (A5), hence from (A7) we have

$$\int_S \Phi_{,j}T^{3j}dS=0. \quad (A8)$$

But from (2.21), the corresponding equations for $\bar{\tau}^{3j}$, and (A6), it is noticed that

$$\left.\begin{aligned} T^{31} &= \frac{p\theta}{A^2}\,\Phi_\xi + \frac{r\theta}{AB}\,\Phi_\eta\,, \\ T^{32} &= \frac{r\theta}{AB}\,\Phi_\xi + \frac{q\theta}{B^2}\,\Phi_\eta\,, \end{aligned}\right\} \tag{A9}$$

so that, on rearrangement, (A8) becomes

$$\int_S \left\{ \frac{p}{A^2}\left(\Phi_\xi + \frac{rA}{pB}\,\Phi_\eta\right)^2 + \frac{pq-r^2}{pB^2}\,\Phi_\eta^2 \right\} dS = 0\,. \tag{A10}$$

But A^2, B^2, p, $pq-r^2$ are positive quantities, so that

$$\Phi_\xi = \Phi_\eta = 0 \text{ in } S\,,$$

and thus $\Phi=$constant in S, or by use of (A6)

$$\Phi = \phi - \bar{y}x + \bar{x}y + \text{constant} \tag{A11}$$

in S.

The displacement $\bar{u}_i$ of (A2) differs from u_i of (A1) by a rigid body displacement, and the stress field τ^{3i} of (2.21) is unchanged in S. So also is σ_{3i} of (3.7), and thus from (4.1) the moment τ of the torsional couple is invariant under a parallel translation of the axis of twist of the cylinder.

REFERENCES

[1] R. K. Brown and E. E. Jones, On the torsion of a curvilinearly aeolotropic cylinder, *Quart. Appl. Math.* 26, pp. 273–275, 1968.
[2] J. E. Adkins, Finite deformation of materials exhibiting curvilinear aeolotropy, *Proc. Roy. Soc.* A 229, pp. 119–134, 1955.
[3] S. G. Lekhnitskii, *Theory of elasticity of an anisotropic elastic body*, Holden-Day, San Francisco, 1963; translated by P. Fern.
[4] A. E. Green and J. E. Adkins, *Large elastic deformations*, Oxford Univ. Press, 1960.
[5] A. E. Green and W. Zerna, *Theoretical elasticity*, Oxford Univ. Press, 1968.
[6] I. S. Sokolnikoff, *Mathematical theory of elasticity*, McGraw-Hill, New York, 1956.
[7] R. M. Morris, The internal problems of two-dimensional potential theory, *Math. Ann.* 116, pp. 374–400, 1939.